Some Problems Relevant to Math 2E Midterm (Spring 2016)

Midterm is on Monday May 9th

As always - quizzes and homeworks are your best study materials!
These questions are mainly to test yourself.

1. (a) Let D be the triangle bounded by the lines $x = 1, y = 2$ and $2x + y = 2$. Give the two expressions for the double integral $\iint\limits_D e^{x-y} dA$ as an iterated integral. (In other words, the expressions using the two orders of integration).

(b) Compute the above double integral (by any method).

2. (a) Let D be the region bounded above by the parabola $y = 2x - x^2$ and below by the line $y = 0$. Give the two expressions for the double integral $\iint\limits_D xy dA$ as an iterated integral.

(b) Evaluate the integral (by any method).

3. (a) Calculate $\int_0^1 \int_{x^{2/3}}^1 x \cos(y^4) dy dx$. Hint: Change the order of integration.

(b) Calculate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{2016} dx dy$.

4. Find the area enclosed by the curve $x^2 + xy + y^2 = 1$.
Hint: Use the substitution $x = u + v\sqrt{3}$, $y = u - v\sqrt{3}$.

5. Evaluate $\int_{z=1}^2 \int_{y=0}^z \int_{x=y}^z xyz \; dx dy dz$.

6. Evalaute $\int_C xy dx + \ln(x) dy$ where C is the curve $x = e^t, \; y = e^{-t}$ from $0 \le t \le 1$.

7. If f is a continuous function defined on $[0, 1]$, show that

$$\int_{x=0}^1 \int_{y=x^2}^x x f(y) dy dx = \frac{1}{2} \int_{y=0}^1 (y - y^2) f(y) dy.$$

Hint: First change the order of integration.

8. Write the expression that would compute $\int_C \mathbf{F} \cdot \mathbf{dr}$ where $\mathbf{F}(x, y) = < x^2y^5, \frac{5}{3}x^3y^4 >$ and C is the path $x = \sin(t), \; y = 1 + \sin(2t)$ from $-\pi/2 \le t \le \pi/2$.
(Don't compute it - the computation is more of a Chapter 16.3 thing).

9. (a) First, express the integral

$$\int_{x=1}^2 \int_{y=\frac{x}{2}}^x \frac{x}{y^2} \sin\left(\frac{\pi x}{y}\right) dy dx$$

as an integral in new variables u, v related to x, y by the equations $x = u, \; y = u/v$. (Take for granted that the above equations is an appropriate change of coordinates, i.e. it is a 1 to 1 mapping on its domain).

(b) Evaluate the integral in (a) by any method.

10. See Section 16.1 Problems 11-14 and 29-32. Try to sketch some of the vector fields, and then match them afterwards. This is because on the exam, if this topic is covered, you will probably have to sketch a vector field instead of just match them.

11. (Stewart 15.8.26 Tweaked) Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\ dV$ where E lies above the cone $z=\sqrt{3(x^2+y^2)}$, and between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.
In fact, all the problems in 15.8.21-27, 41-43 are good exercises on Spherical coordinates.

12. (Stweart 15.7.20) Evaluate $\iiint_E (x-y)dV$ where E is the solid between the cylinders $x^2+y^2=1$, $x^2+y^2=16$, above the xy-plane, and below the plane $z=y+4$.
In fact, all the problems 15.7.17-25, 29-30 are good examples on Cylindrical coordinates.

13. (a) Find the length of the curve from $0 \le t \le 10$,

$$\mathbf{r}(t) = <\cos(t)+t\sin(t),\ \sin(t)-t\cos(t)>$$

(b) Let the curve C be the above parameterization from $0 \le t \le 10$. Compute $\int_C x\ ds$.

14. (a) Compute $\int_C (x^2y+\sin x)dy$ where C is the line from $(1,0)$ to $(0,0)$ followed by the arc of the parabola $y=x^2$ from (0,0) to (π,π^2).
(b) Compute the same integral, but if C is the line from $(0,1)$ to $(0,0)$ along with the arc of the parabola $y=x^2$ from $(0,0)$ to (π,π^2).
(c) Compute the same integral, but C is the (backwards) line $y=x$ from (1,1) to (0,0) along with the arc of the parabola $y=x^2$ from $(0,0)$ to (π,π^2).

15. #14 is $\sim$ 16.2.5. For other line integral problems, see 16.2.1-16, 19-22 (and the homework).

16. Let $A(q)$ be the area of the region $R_q := \{(x,y)$ such that $x \ge 0, y \ge 0$, and $\sqrt{x}+\sqrt{y} \le q\}$.
* This one may seem a bit weird. It is a neat problem, but it's very unlike your homework *
* At least do part (d), but you should attempt parts (a)-(c).*
(a) Consider the transformation $x=pu, y=pv$ where p is a number/constant. Find the number p, which depends on q, that takes the region S in the uv-plane defined by
$S := \{(u,v)$ such that $u \ge 0,\ v \ge 0$ and $\sqrt{u}+\sqrt{v} \le 1\}$ to the given region R_q in the xy-plane.
(b) Use the transformation in (a) to write $A(q)$ as a double integral over the region S. i.e. since

$$A(q) := \iint_{R_q} 1 \cdot dxdy = \iint_S 1 \cdot J\ dudv,$$

find the integral on the right hand side.
(c) Using (b), what happens to $A(q)$ when q is doubled?
(d) Use the change of variables $x=r^2, y=s^2$ to find $A(q)$.

$$\left\{\begin{array}{l|l}
\text{Spherical Coordinates} & \text{Cylidrical Coodrinates} \\
x=\rho\sin\phi\cos\theta, \quad \rho=\sqrt{x^2+y^2+z^2} & x=r\cos\theta, \quad r=\sqrt{x^2+y^2}. \\
y=\rho\sin\phi\sin\theta, \quad \theta=\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right), or\ \tan^{-1}\left(\frac{y}{x}\right) & y=r\sin\theta, \quad \theta=\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right), or\ \tan^{-1}\left(\frac{y}{x}\right) \\
z=\rho\cos\phi, \quad \phi=\cos^{-1}\left(\frac{z}{\rho}\right), or\ \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) & z=z \\
dV=\rho^2\sin\phi\ d\rho d\theta d\phi, \text{ The Jacobian } J=\rho^2\sin\phi & dV=r\ dzdrd\theta, \text{ The Jacobian } J=r
\end{array}\right.$$

** Be careful when using the Inverse Trigs, that they make physical sense! **